A NEW PROOF OF THE COMPLETENESS OF THE ŁUKASIEWICZ AXIOMS[1]

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The purpose of this note is to provide a new proof for the completeness of the Łukasiewicz axioms for infinite valued propositional logic. For the existing proof of completeness and a history of the problem in general we refer the readers to [1; 2; 3; 4]. The proof as was given in [4] was essentially metamathematical in nature; the proof we offer here is essentially algebraic in nature, which, to some extent, justifies the program initiated by the author in [2].

In what follows we assume thorough familiarity with the contents of [2] and adopt the notation and terminology of [2]. The crux of this proof is contained in the following two observations: Instead of using locally finite MV-algebras as the basic building blocks in the structure theory of MV-algebras, we shall use linearly ordered ones. The one-to-one correspondence between linearly ordered MV-algebras and segments of ordered abelian groups enables us to make use of some known results in the first-order theory of ordered abelian groups[2].

We say that P is a *prime* ideal of an MV-algebra A if, and only if, (i) P is an ideal of A, and (ii) for each $x, y \in A$, either $x\bar{y} \in P$ or $\bar{x}y \in P$.

LEMMA 1. *If P is a prime ideal of A, then A/P is a linearly ordered MV-algebra.*

Proof. By 3.11 of [2], we have to prove that given x/P and y/P, either $x/P \leqq y/P$ or $y/P \leqq x/P$. But by 1.13 of [2], this just means that either $x\bar{y} \in P$ or $\bar{x}y \in P$.

LEMMA 2. *If $a \in A$ and $a \neq 0$, then there exists a prime ideal P of A such that $a \notin P$.*

Proof. Consider an ideal I of A which is maximal with respect to the property that $a \notin I$. We show that I is a prime ideal. Let $x, y \in A$ and assume $x\bar{y} \notin I$ and $\bar{x}y \notin I$. Thus the ideal generated by I and the element $x\bar{y}$ would contain the element a, i.e.,

$$(1) \qquad a \leqq t + p(x\bar{y}) \qquad \text{for some } t \in I \text{ and } p \text{ integer.}$$

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[2] The author wishes to give thanks to Dana Scott who first suggested this new angle of attack; in particular, Scott has simplified the original argument of the author for Lemma 2.

Similarly, the ideal generated by I and $\bar{x}y$ would also contain the element a, i.e.,

$$(2) \qquad a \leqq s + q(\bar{x}y) \qquad \text{for some } s \in I \text{ and } q \text{ integer.}$$

Let $u = s+t$ and let $n = \max(p, q)$. Then clearly $u \in I$ and from (1) and (2),

$$(3) \qquad a \leqq u + n(x\bar{y}) \quad \text{and} \quad a \leqq u + n(\bar{x}y).$$

From (3) and Axiom 11 of [2],

$$(4) \qquad a = a \wedge a \leqq [u + n(x\bar{y})] \wedge [u + n(\bar{x}y)] = u + [n(x\bar{y}) \wedge n(\bar{x}y)].$$

Now, using the dual of 3.7 of [2], we see that

$$n(x\bar{y}) \wedge n(\bar{x}y) = 0.$$

Thus, by (4), $a \leqq u$ which implies the contradiction that $a \in I$.

LEMMA 3. *Every MV-algebra is a subdirect product of linearly ordered MV-algebras.*

Proof. This is obvious by Lemma 2, as all we need to show is that the set intersection of all prime ideals of an MV-algebra contains the 0 element only.

Given an additive ordered abelian group G (with the operations $+$ and $-$, the identity 0, and the ordering $\leqq$) let the *segment* $G[c]$ determined by a positive element c of G be the set of all elements $x \in G$ such that $0 \leqq x \leqq c$. We define the operations $+'$, $'$, and $^{-'}$ on the elements of $G[c]$ as follows:

$$x +' y = \min(c, x + y),$$

$$\bar{x}' = c - x,$$

$$x'y + (\bar{x}' =' \bar{y}')^{-'}.$$

LEMMA 4. *The algebraic system determined by the set $G[c]$, the operations defined above, and the distinguished elements* 0 *and* c *is a linearly ordered MV-algebra.*

Proof. The proof consists in checking that all axioms of MV-algebras hold in $G[c]$, plus the fact that it is linearly ordered. We shall not give the details here.

What we now wish to establish is the converse to Lemma 4. Given an MV-algebra A, we let A^* be the set of all ordered pairs (m, x) where m is an integer and $x \in A$. On the set A^* we define the following:

$$(m + 1, 0) = (m, 1),$$

$$(m, x) + (n, y) = (m + n, x + y) \qquad \text{if } x + y < 1,$$

$$(m, x) + (n, y) = (m + n + 1, xy) \qquad \text{if } x + y = 1,$$

$$-(m, x) = (-m - 1, \bar{x}).$$

LEMMA 5. *Let A be a linearly ordered MV-algebra, then the set A^* under the operations $+$ and $-$ and with the distinguished element* $(0, 0)$ *is an additive ordered abelian group.*

Proof. We first prove a result which belongs to the elementary theory of MV-algebras:

(1) If x, y, z are elements of a linearly ordered MV-algebra and if $\bar{x} \leqq y$ and $y+z<1$, then $x(y+z)=xy+z$. This can be seen as follows:

$$xy + z + \bar{x} = xy + \bar{x} + z = (y \vee \bar{x}) + z = y + z,$$

and

$$x(y + z) + \bar{x} = \bar{x} \vee (y + z) = y + z.$$

Therefore,

$$xy + z + \bar{x} = x(y + z) + \bar{x}.$$

Since $y+z<1$, the conclusion of (1) follows by 3.13 of [2].

To proceed with the main proof, we first check that the definitions of $+$ and $-$ as given above is consistent with respect to the equality $(m+1, 0) = (m, 1)$. Also, it is clear that the operation $+$ on A^* is commutative and that each element of A^* has an additive inverse. It now remains to prove that $+$ is associative. Therefore, let three elements (m, x), (n, y), (q, z) of A^* be given. We wish to show that

$$(m, x) + [(n, y) + (q, z)] = [(m, x) + (n, y)] + (q, z). \tag{2}$$

We proceed by cases.

CASE 1. $x+y+z<1$. It is clear that $x+y<1$ and $y+z<1$, therefore (2) becomes

$$(m + (n + q), x + (y + z)) = ((m + n) + q, (x + y) + z)$$

which certainly holds.

CASE 2. $x+y+z=1$. There are now four subcases.

CASE 2a. $x+y<1$ and $y+z<1$. In this case (2) becomes

$$(m + n + q + 1, x(y + z)) = (m + n + q + 1, (x + y)z). \tag{3}$$

Suppose $x+z=1$. Then $\bar{x} \leqq z$ and $\bar{z} \leqq x$, and by (1),

$$x(y + z) = y + xz = (x + y)z$$

which proves (3). Suppose now

$$x + z < 1. \tag{4}$$

Since $(x+y)^- \leqq z$,

$$z = z \vee (x + y)^- = (x + y)z + (x + y)^-,$$

and

$$z + x = (x + y)z + (x + y)^{-} + x = (x + y)z + \bar{x}\bar{y} + x = (x + y)z + xy + \bar{y}.$$

Since $x+y<1$, we have that $xy=0$, therefore

$$z + x = (x + y)z + \bar{y}. \tag{5}$$

Similarly, as $(y+z)^{-} \leqq x$, we obtain (using the fact $y+z<1$)

$$\begin{aligned} z + x &= z + (y + z)x + (y + z)^{-} = x(y + z) + z + \bar{y}\bar{z} \\ &= x(y + z) + yz + \bar{y} = x(y + z) + \bar{y}. \end{aligned} \tag{6}$$

(4), (5), (6), and 3.13 of [2] enable us to cancel $\bar{y}$ and obtain (3).

CASE 2b. $x+y<1$ and $y+z=1$. In this case the right hand side of (2) becomes

$$(m + n + q + 1, (x + y)z) \tag{7}$$

and the left hand side of (2) becomes

$$(m, x) + (n + q + 1, yz). \tag{8}$$

Since $x+y<1$, hence $x+yz<1$, therefore (8) becomes

$$(m + n + q + 1, x + yz). \tag{9}$$

Using (1), we see easily that

$$(x + y)z = x + yz,$$

hence the equality of (7) and (9) is assured.

CASE 2c. $x+y=1$ and $y+z<1$. The argument for this case is analogous to that of Case 2b.

CASE 2d. $x+y=1$ and $y+z=1$. In this case the right hand side of (2) becomes

$$(m + n + 1, xy) + (q, z) \tag{10}$$

and the left hand side of (2) becomes

$$(m, x) + (n + q + 1, yz). \tag{11}$$

We consider two more subcases.

CASE 2d(i). $xy+z=1$. In this case we show that $x+yz=1$. We have that $\bar{x}+\bar{y} \leqq z$ and $\bar{x} \leqq y$, thus

$$\bar{x} = \bar{x} \wedge y = (\bar{x} + \bar{y})y \leqq zy. \tag{12}$$

(12) of course implies that $x+yz=1$, hence both (10) and (11) are equal to $(m+n+q+2, xyz)$ which proves (2).

CASE 2d(ii). $xy+z<1$. In this case by considering the argument in Case 2d(i) and symmetry, we also have $x+yz<1$. Hence (10) becomes

$$(m + n + q + 1, xy + z)$$

and (11) becomes

$$(m + n + q + 1, x + yz).$$

We now have to show under these conditions,

$$xy + z = x + yz. \tag{13}$$

Since $xy+z<1$, $x+yz<1$, $\bar{y}\leqq z$, and $\bar{y}\leqq x$, we have by (1)

$$(xy + z)y = xy + zy,$$
$$(x + yz)y = xy + zy,$$

and

$$(xy + z)y = (x + yz)y. \tag{14}$$

Adding $\bar{y}$ to both sides of (14), we get by using the commutativity of $\vee$,

$$\bar{y}\bar{z}(\bar{x} + \bar{y}) + xy + z = \bar{y}\bar{x}(\bar{y} + \bar{z}) + x + yz. \tag{15}$$

But since $x+y=y+z=1$, $\bar{y}\bar{z}=\bar{x}\bar{z}=0$, therefore (15) leads to the desired equality (13).

Finally, in order to show that A^* is an ordered group, we simply exhibit the ordering relation $\leqq$ and leave it to the reader to check that the ordering is preserved by the group operations:

$$(m, x) \leqq (n, y) \text{ if and only if either } m < n \text{ or } m = n \text{ and } x \leqq y.$$

Lemma 6. *If A is a linearly ordered MV-algebra, then $A^*[(0, 1)]$ is isomorphic with A; furthermore, the element $(0, 1)$ in A^* has the property that for each $x\in A^*$, there exists an n such that $n(0, 1)\leqq x\leqq (n+1)(0, 1)$. On the other hand, if G is an ordered abelian group and c is a positive element of G such that for each $x\in G$, there exists an n such that $nc\leqq x<(n+1)c$, then $G[c]^*$ is isomorphic with G.*

Proof. The first part of the lemma is clearly true from our construction of A^*. For the second part we shall exhibit the isomorphism of G onto $G[c]^*$. For each $x\in G$, there exists an n_x such that $n_x c\leqq x<(n_x+1)c$. The function f is defined as follows:

$$f(x) = (n_x, x - n_x c).$$

It is an elementary exercise to prove that f is well-defined and is an isomorphism.

Incidentally, we remark here that for Lemmas 6 and 7, if A is a locally finite MV-algebra then A^* is an Archimedean ordered abelian group. Using this fact we see that the conjecture stated after 3.21 of [2] is true. It also follows that every locally finite MV-algebra has at most a continuum number of elements.

LEMMA 7. *To each identity E in the theory of MV-algebras, there corresponds an universal sentence E^* (with one free variable c) in the theory of ordered abelian groups such that for any linearly ordered MV-algebra A, E holds in A if and only if E^* holds in A^* with the free variable c interpreted as the element* $(0, 1)$.

Proof (*in outline*). Given an identity E in the theory of MV-algebras, we assume that $x_1, x_2, \cdots, x_n$ are the only variables occurring in E and that the identity E is built up from the variables, the constants 0 and 1, and the operations $+$ and $^-$. We arrive at the associated universal sentence E^* in a finite number of steps in the following manner: First we replace in E the symbol 1 by the symbol c. Then we replace (in the order of their lengths) each expression of the form

$$\nu + \xi$$

in E by the expression

$$\min (\nu +^* \xi, c),$$

and each expression of the form

$$\bar{\xi}$$

in E by the expression

$$c - \xi.$$

Thus we obtain, at the end of the process, an expression E' which is built up from the group operations $+^*$ and $-$ and the function $\min (x, y)$. Let now E'' be the expression obtained from E' by simply removing everywhere in E' the symbol $*$. Finally, the universal sentence E^* in the theory of ordered abelian groups is

$$E^* = (x_1) \cdots (x_n)(0 \leqq x_1 \leqq c \wedge \cdots \wedge 0 \leqq x_n \leqq c \rightarrow E'').$$

From our construction of E^* and A^* it is evident that E holds in A if and only if E^* holds in A^* with c interpreted as $(0, 1)$.

At this point we shall make use of two known results:

(I) Every ordered abelian group can be embedded in a divisible ordered abelian group.

(II) The first-order theory of divisible ordered abelian groups is complete.

Result (I) is well-known, and result (II) can be found in [5] and [6][3]. From (I) and (II) we infer immediately that

(III) An universal sentence ξ in the first-order theory of ordered abelian groups holds in the additive group R of rationals if and only if ξ holds in every ordered abelian group.

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[3] Indeed, (II) is a result of Tarski's which somehow never appeared explicitly as such in print. The closest reference to it can be found in [5] and in an English translation of [5] in [6, p. 134], second paragraph.

We are now ready for

LEMMA 8. *An identity E (in the theory of MV-algebras) holds in the linearly ordered MV-algebra $R[1]$ if and only if it holds in every linearly ordered MV-algebra.*

Proof. The lemma is trivial in one direction. Assume now an identity E is given which does not hold in some linearly ordered MV-algebra A. Thus E^* will not hold in A^* with c interpreted as the element $(0, 1)$; in particular, the universal sentence (without free variables)

$$\xi = (c)(0 < c \rightarrow E^*)$$

will not hold in A^*. By result (III), ξ does not hold in the group R, i.e.,

(1) there exists an element (positive) c in R such that E^* does not hold in R.

By the fact that there is an automorphism (both group and order) of the rationals R onto R mapping c onto 1, we see from (1) that E^* does not hold in R with c interpreted as 1. By Lemma 6, $R[1]^*$ is isomorphic with R, hence we finally arrive at the result that E does not hold in $R[1]$ which proves the lemma.

THEOREM. *In the Lukasiewicz axiom system for infinitely valued propositional logic every valid formula is provable.*

Proof. From our previous results and considerations to be found in §5 of [2], we only need to show that every identity E which holds in the linearly ordered MV-algebra $R[1]$ holds in the algebra L. By Lemma 3, L is a subalgebra of a direct product of linearly ordered algebras. By Lemma 8, if E holds in $R[1]$, then E holds in each one of these linearly ordered factors; which, of course, implies that E holds in L.

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